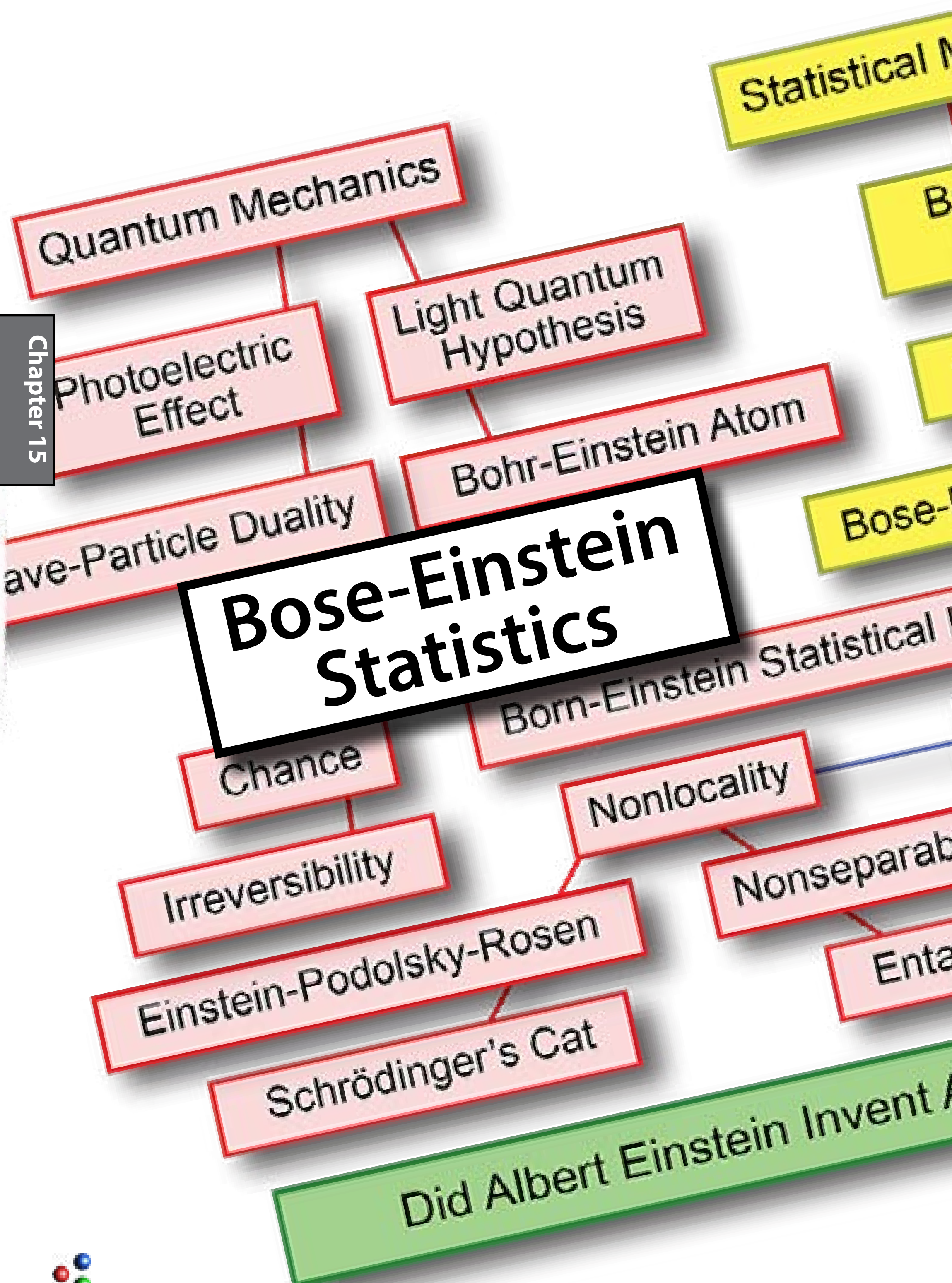
Chapter 15

Bose-Einstein Statistics



Bose-Einstein Statistics

In 1924, Einstein received an amazing very short paper sent from India by Satyendra Nath Bose. Einstein must have been pleased to read the title, "Planck's Law and the Hypothesis of Light Quanta." It was more attention to Einstein's 1905 work than anyone had paid in nearly twenty years. The paper began by claiming that the "phase space" (a combination of 3-dimensional coordinate space and 3-dimensional momentum space) should be divided into small volumes of h^3, the cube of Planck's constant. By counting the number of possible distributions of light quanta over these cells, Bose claimed he could calculate the entropy and all other thermodynamic properties of the radiation.

Bose easily derived Planck's inverse exponential function $1/(e^{h\nu/kT}-1)$. Einstein too had derived this. Maxwell and Boltzmann derived the so-called Boltzmann factor $e^{-h\nu/kT}$, by analogy from the Gaussian exponential tail of probability and the theory of errors.

Max Planck had simply guessed this expression from Wien's radiation distribution law $ae^{-b\nu/T}$ by adding the term - 1 in the denominator of Wien's law in the form $a/e^{b\nu/T}$ to get $1/(e^{h\nu/kT}-1)$.

All previous derivations of the Planck law, including Einstein's of 1916-17 (which Bose called "remarkably elegant"), used classical electromagnetic theory to derive the density of radiation, the number of "modes" or "degrees of freedom" per unit volume of the radiation field,

$$\rho_\nu d\nu = (8\pi\nu^2 d\nu / c^3).$$

Bose considered the radiation to be enclosed in a volume V with total energy E. He assumed that various types of quanta are present with abundances N_i and energy $h\nu_i$ $(i = 0 \text{ to } i = \infty)$.

The total energy is then

$$E = \Sigma_i N_i h\nu_i = V \int \rho_\nu d\nu.$$

But now Bose showed he could get ρ_ν with a simple statistical mechanical argument remarkably like that Maxwell used to derive his distribution of molecular velocities. Maxwell said that the three directions of velocities for particles are independent of one another, and of course equal to the total momentum,

$p_x^{\,2} + p_y 2 + p_z^{\,2} = p^2$,

Bose just used Einstein's relation for the momentum of a photon,

$p = h\nu / c$.

The momentary state of the quantum is characterized by its coordinates x, y, z and the corresponding components of the momentum p_x, p_y, p_z. These six quantities can be considered as point coordinates in a six–dimensional space, where we have the relation

$p_x^{\,2} + p_y^{\,2} + p_z^{\,2} = h^2\nu^2 / c^2$.

This led Bose to calculate a frequency interval in phase space as

$\int dx\, dy\, dz\, dp_x\, dp_y\, dp_z = 4\pi V\, (h\nu / c)3\, (h\, d\nu / c)$

$= 4\pi\, (h^3\, \nu^2 / c^3)\, V\, d\nu$,

Bose simply divided this expression by h^3, multiplied by 2 to account for two polarization degrees of freedom of light, and he had derived the number of cells belonging to $d\nu$,

$\rho_\nu d\nu = (8\pi\nu^2 d\nu / c^3)\, E$,

This expresion is well-known from classical electrodynamics, but Bose found this result without using classical radiation laws, a correspondence principle, or even Wien's law. His derivation was purely statistical mechanical, based only on the number of quantum cells in phase space and the number of ways N photons can be distributed among them.

When Bose calculated the number of ways of placing light quanta in these cells, i.e., the number of cells with no quanta, the number with one, two, three, etc., he put no limits on the number of quanta in a h^3 cell.

Einstein saw that unlimited numbers of particles close together implies extreme densities and low-temperature condensation of any particles with integer values of the spin. Material particles like electrons are known to limit the number of particles in a cell to two, one with spin up, one spin down. They have half-integer spin.

Particles with integer-value spins follow the new Bose-Einstein quantum statistics. This relation between spin and statistics is called the spin-statistics theorem of Wolfgang Pauli.

When identical particles in a two-particle wave function are exchanged, the antisymmetric wave function for fermions changes sign. The symmetric boson wave function does not change sign.

Paul Dirac quickly developed the quantum statistics of half-integer spin particles, now called Fermi-Dirac statistics. A maximum of two particles, with opposite spins, can be found in the fundamental h^3 volume of phase space identified by Bose. This explains why there are a maximum of two electrons in the first electron shell of any atom.

Einstein's discovery led us to "Bose-Einstein condensations" as temperatures approach absolute zero, because there is no limit on the number of integer-spin particles that can be found in an h^3 volume of phase space. This work is frequently attributed to Bose instead of Einstein. Particles with integer spin are called "bosons." In a similar irony, particles with half-integer spin that obey Pauli's exclusion principle are called "fermions."

Einstein's discovery of quantum statistics is often seen as his last *positive* contribution to quantum physics. Few historians point out that Einstein was first to see the two kinds of elementary particles in today's "standard model!"

Einstein's most profound insight into elementary particles might be their indistinguishability, their interchangeability. Particles are not independent of one another, perhaps even when they are apparently far apart, like electrons in a two-particle wave function. See their *entanglement* in chapter 29.